Parametrization of fermion mixing matrices in Kobayashi-Maskawa form

Nan Qin[1] and Bo-Qiang Ma[1,2,*]

[1]*School of Physics and State Key Laboratory of Nuclear Physics and Technology, Peking University, Beijing 100871, China*
[2]*Center for High Energy Physics, Peking University, Beijing 100871, China*

Recent works show that the original Kobayashi-Maskawa (KM) form of fermion mixing matrix exhibits some advantages, especially when discussing problems such as unitarity boomerangs and maximal CP violation hypothesis. Therefore, the KM form of fermion mixing matrix is systematically studied in this paper. Starting with a general triminimal expansion of the KM matrix, we discuss the triminimal and Wolfenstein-like parametrizations with different basis matrices in detail. The quark-lepton complementarity relations play an important role in our discussions on describing quark mixing and lepton mixing in a unified way.

PACS numbers: 12.15.Ff, 14.60.-z, 14.60.Pq, 14.65.-q, 14.60.Lm

I. INTRODUCTION

As it is well known, the mixing between different generations of fermions is one of the most interesting issues in particle physics. For quarks, the mixing matrix is described by the Cabibbo[1]-Kobayashi-Maskawa[2](CKM) matrix V_{CKM}, and in the lepton sector, it is described by the Pontecorvo[3]-Maki-Nakawaga-Sakata[4] (PMNS) matrix U_{PMNS},

$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}, \quad U_{\mathrm{PMNS}} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}.$$

Before more underlying theory of the origin of the mixing is found, parametrizing the mixing matrices properly is helpful to understanding the mixing pattern and search for deviations from the standard model both theoretically and experimentally. A commonly used form of the fermion mixing matrix is the standard parametrization proposed by Chau and Keung (CK) [5]

$$\begin{aligned} V_{\mathrm{CK}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{\mathrm{CK}}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{\mathrm{CK}}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{\mathrm{CK}}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{\mathrm{CK}}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{\mathrm{CK}}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{\mathrm{CK}}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{\mathrm{CK}}} & c_{23}c_{13} \end{pmatrix}, \end{aligned} \quad (1)$$

where $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$ $(i,j = 1,2,3)$ are the rotation angles, and δ_{CK} is the CP-violating phase in the CK parametrization.

Recently, it has been pointed out in many works [6–8] that the original Kobayashi-Maskawa (KM) [2] matrix is convenient when discussing problems such as unitarity boomerangs [6] and maximal CP violation hypothesis [9]. The original KM mixing matrix is given by

$$\begin{aligned} V_{\mathrm{KM}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & -s_2 \\ 0 & s_2 & c_2 \end{pmatrix} \begin{pmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & e^{i\delta_{\mathrm{KM}}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & s_3 \\ 0 & s_3 & -c_3 \end{pmatrix} \\ &= \begin{pmatrix} c_1 & -s_1 c_3 & -s_1 s_3 \\ s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta_{\mathrm{KM}}} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta_{\mathrm{KM}}} \\ s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta_{\mathrm{KM}}} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta_{\mathrm{KM}}} \end{pmatrix}, \end{aligned} \quad (2)$$

in which $s_i = \sin\theta_i$, $c_i = \cos\theta_i$ $(i = 1,2,3)$ are Euler angles, and δ_{KM} is the CP-violating phase in the KM parametrization.

---

*Electronic address: mabq@pku.edu.cn

If neutrinos are of the Majorana type, there should be an additional diagonal matrix with two Majorana phases $P = \text{diag}(e^{i\alpha_1/2}, e^{i\alpha_2/2}, 1)$ multiplied to Eqs. (2) and (1) from the right. In this paper, we consider the neutrinos as Dirac neutrinos, and the presentation of formalisms for Majorana neutrinos can be derived straightforwardly by including the additional phases. In the following, we omit the subscript CK and KM since we deal with only the KM form. We also denote parameters in the quark sector with superscript Q and in the lepton sector with superscript L if necessary (except for the CP-violating phases in expressions with the consideration of concinnity, and bearing in mind that δ^Q appears in V_{CKM} while δ^L appears in U_{PMNS}).

The magnitudes of the CKM matrix elements have been well determined with [10]

$$\begin{pmatrix} 0.97428 \pm 0.00015 & 0.2253 \pm 0.0007 & 0.00347^{+0.00016}_{-0.00012} \\ 0.2252 \pm 0.0007 & 0.97345^{+0.00015}_{-0.00016} & 0.0410^{+0.0011}_{-0.0007} \\ 0.00862^{+0.00026}_{-0.00020} & 0.0403^{+0.0011}_{-0.0007} & 0.999152^{+0.000030}_{-0.000045} \end{pmatrix} . \tag{3}$$

With $|V_{ud}|$, $|V_{ub}|$, $|V_{td}|$ and $|V_{cb}|$ as input parameters, one can easily get the ranges for the angle parameters as

$$\theta_1^Q = 0.2273^{+0.0011}_{-0.0003}, \quad \theta_2^Q = 0.0383^{+0.0011}_{-0.0010}, \quad \theta_3^Q = 0.0154^{+0.0008}_{-0.0006}, \quad \delta^Q = 90.33°^{+2.85°}_{-4.57°} . \tag{4}$$

The last equation apparently implies that the KM phase convention is consistent with the maximal CP violation hypothesis.

For lepton mixing, the ranges for the PMNS matrix elements have been also constrained by (at 3σ level) [11]

$$\begin{pmatrix} 0.77 - 0.86 & 0.50 - 0.63 & 0.00 - 0.22 \\ 0.22 - 0.56 & 0.44 - 0.73 & 0.57 - 0.80 \\ 0.21 - 0.55 & 0.40 - 0.71 & 0.59 - 0.82 \end{pmatrix} . \tag{5}$$

Since the data are not accurate enough here, we do not calculate the parameters for leptons as what we do for quarks. Instead, the numerical results are presented in Sec. III and Sec. IV where unified description of quark mixing and lepton mixing are discussed.

When studying mixing, it is useful to parametrize the matrix according to the hierarchical structure of the mixing to reveal more physical information about the underlying theory. The Wolfenstein parametrization for quarks is a famous example of this type, where V_{CKM} is parametrized as [12]

$$V_{\text{CKM}} = \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4) . \tag{6}$$

The up-to-date fit for the Wolfenstein parameters gives [10]

$$\lambda = 0.2253 \pm 0.0007 , \qquad A = 0.808^{+0.022}_{-0.015} ,$$
$$\rho(1 - \lambda^2/2 + \ldots) = 0.132^{+0.022}_{-0.014} , \quad \eta(1 - \lambda^2/2 + \ldots) = 0.341 \pm 0.013 . \tag{7}$$

The Wolfenstein parametrization is actually an expansion of V_{CKM} around the unit matrix basis with λ as the expanding parameter. In this type of parametrization, the choice of the parameters and where to put them are arbitrary, making the meaning of the parameters subtle to some extent. For example, the CP-violating phase δ is not independent, i.e., it is determined by two parameters η and ρ with $\tan\delta = \eta/\rho$. Therefore it would be better to expand the mixing matrix using small parameters with explicit physical meaning. A good choice is the idea of triminimal parametrization [13] with an approximation as the basis matrix to the lowest order. The triminimal expansion of the quark and lepton mixing pointed out a new way to parametrize the mixing matrix with all angle parameters small, and with the CP-violating phase parameter free from others. The parameters are completely determined when the basis matrix is chosen.

For quark mixing, the unit matrix is very simple while the matrix suggested in Ref. [14] given by

$$V_0' = \begin{pmatrix} \frac{\sqrt{2}+1}{\sqrt{6}} & \frac{\sqrt{2}-1}{\sqrt{6}} & 0 \\ -\frac{\sqrt{2}-1}{\sqrt{6}} & \frac{\sqrt{2}+1}{\sqrt{6}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8}$$

is more close to experimental data so that they are both good choices.

In lepton sector, it has been common to choose the bimaximal matrix [15] and/or the tri-bimaximal matrix [16] as the basis matrices

$$U_{\rm bi} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & -1/2 & 1/\sqrt{2} \end{pmatrix}, \quad U_{\rm tri} = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}. \tag{9}$$

Although the former one is not favored by present experimental data as the later one, it looks more symmetric with also a possible connection with the unit basis in quark mixing [17]. The tri-bimaximal basis is very close to experimental data and can serve as a good approximation for lepton mixing.

Although it seems that the mixing of quarks and leptons are unrelated with each other, there indeed exist phenomenological relations between mixing angles called quark-lepton complementarity (QLC) [18]. For KM parameters, the QLC relations still stand [7, 19],

$$\theta_1^Q + \theta_1^L = \frac{\pi}{4}, \quad \theta_2^Q + \theta_2^L = \frac{\pi}{4}, \quad \theta_3^Q \sim \theta_3^L \sim 0. \tag{10}$$

It has been discussed that the quark mixing matrix and the lepton mixing matrix can be parametrized in a unified way with the QLC relations [20]. However, the discussions in Ref. [20] are based on the CK phase convention. Since the KM form of mixing matrices is promoted in many works, a detailed study of it is necessary, and unified parametrizations in KM phase convention may be helpful in both theoretical and phenomenological studies.

The outline of this paper is as follows. In Sec. II, the general expressions of triminimal expansion of the KM matrix are presented. In Sec. III, we study the triminimal parametrization with unit matrix and bimaximal matrix as the basis for quarks and leptons respectively. Wolfenstein-like parametrizations are also discussed and numerical results of the parameters are presented. In Sec. IV, triminimal and Wolfenstein-like parametrizations are discussed in tri-bimaximal pattern. We show that the expansions converge much faster in both quark sector and lepton sector. In both Sec. III and Sec. IV, QLC relations play an important role in our discussions of parametrizing $V_{\rm CKM}$ and $U_{\rm PMNS}$ in a unified way. Finally, we present our conclusions in Sec. V.

II. THE GENERAL RESULTS OF TRIMINIMAL EXPANSION OF KM MATRIX

The idea of the triminimal parametrization [13] is to express a mixing angle in the mixing matrices as the sum of a zeroth order angle θ^0 and a small perturbation angle ϵ as

$$\theta_1 = \theta_1^0 + \epsilon_1, \quad \theta_2 = \theta_2^0 + \epsilon_2, \quad \theta_3 = \theta_3^0 + \epsilon_3. \tag{11}$$

With the deviations ϵ_i, one can expand the mixing matrices in powers of ϵ_i while different choices of θ_i^0 lead to different matrices as the zeroth order of the expansion. Generally, to the second order of ϵ_i, the mixing matrix is expanded as

$$
\begin{aligned}
V =& \begin{pmatrix} c_1^0 & -s_1^0 c_3^0 & -s_1^0 s_3^0 \\ s_1^0 c_2^0 & c_1^0 c_2^0 c_3^0 - s_2^0 s_3^0 e^{i\delta} & c_1^0 c_2^0 s_3^0 + s_2^0 c_3^0 e^{i\delta} \\ s_1^0 s_2^0 & c_1^0 s_2^0 c_3^0 + c_2^0 s_3^0 e^{i\delta} & c_1^0 s_2^0 s_3^0 - c_2^0 c_3^0 e^{i\delta} \end{pmatrix} + \epsilon_1 \begin{pmatrix} -s_1^0 & -c_1^0 c_3^0 & -c_1^0 s_3^0 \\ c_1^0 c_2^0 & -c_2^0 c_3^0 s_1^0 & -c_2^0 s_1^0 s_3^0 \\ c_1^0 s_2^0 & -c_3^0 s_1^0 s_2^0 & -s_1^0 s_2^0 s_3^0 \end{pmatrix} \\
&+ \epsilon_2 \begin{pmatrix} 0 & 0 & 0 \\ -s_1^0 s_2^0 & -c_1^0 c_3^0 s_2^0 - c_2^0 s_3^0 e^{i\delta} & -c_1^0 s_2^0 s_3^0 + c_2^0 c_3^0 e^{i\delta} \\ c_2^0 s_1^0 & c_1^0 c_2^0 c_3^0 - s_2^0 s_3^0 e^{i\delta} & c_1^0 c_2^0 s_3^0 + c_3^0 s_2^0 e^{i\delta} \end{pmatrix} \\
&+ \epsilon_3 \begin{pmatrix} 0 & s_1^0 s_3^0 & -c_3^0 s_1^0 \\ 0 & -c_1^0 c_2^0 s_3^0 - c_3^0 s_2^0 e^{i\delta} & c_1^0 c_2^0 c_3^0 - s_2^0 s_3^0 e^{i\delta} \\ 0 & -c_1^0 s_2^0 s_3^0 + c_2^0 c_3^0 e^{i\delta} & c_1^0 c_3^0 s_2^0 + c_2^0 s_3^0 e^{i\delta} \end{pmatrix} + \frac{1}{2}\epsilon_1^2 \begin{pmatrix} -c_1^0 & c_3^0 s_1^0 & s_1^0 s_3^0 \\ -c_2^0 s_1^0 & -c_1^0 c_2^0 c_3^0 & -c_1^0 c_2^0 s_3^0 \\ -s_1^0 s_2^0 & -c_1^0 c_3^0 s_2^0 & -c_1^0 s_2^0 s_3^0 \end{pmatrix} \\
&+ \frac{1}{2}\epsilon_2^2 \begin{pmatrix} 0 & 0 & 0 \\ -c_2^0 s_1^0 & -c_1^0 c_2^0 c_3^0 + s_2^0 s_3^0 e^{i\delta} & -c_1^0 c_2^0 s_3^0 - c_3^0 s_2^0 e^{i\delta} \\ -s_1^0 s_2^0 & -c_1^0 c_3^0 s_2^0 - c_2^0 s_3^0 e^{i\delta} & -c_1^0 s_2^0 s_3^0 + c_2^0 c_3^0 e^{i\delta} \end{pmatrix} \\
&+ \frac{1}{2}\epsilon_3^2 \begin{pmatrix} 0 & c_3^0 s_1^0 & s_1^0 s_3^0 \\ 0 & -c_1^0 c_2^0 c_3^0 + s_2^0 s_3^0 e^{i\delta} & -c_1^0 c_2^0 s_3^0 - c_3^0 s_2^0 e^{i\delta} \\ 0 & -c_1^0 c_3^0 s_2^0 - c_2^0 s_3^0 e^{i\delta} & -c_1^0 s_2^0 s_3^0 + c_2^0 c_3^0 e^{i\delta} \end{pmatrix} + \epsilon_1 \epsilon_2 \begin{pmatrix} 0 & 0 & 0 \\ -c_1^0 s_2^0 & c_3^0 s_1^0 s_2^0 & s_1^0 s_2^0 s_3^0 \\ c_1^0 c_2^0 & -c_2^0 c_3^0 s_1^0 & -c_2^0 s_1^0 s_3^0 \end{pmatrix} \\
&+ \epsilon_2 \epsilon_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_1^0 s_2^0 s_3^0 - c_2^0 c_3^0 e^{i\delta} & -c_1^0 c_3^0 s_2^0 - c_2^0 s_3^0 e^{i\delta} \\ 0 & -c_1^0 c_2^0 s_3^0 - c_3^0 s_2^0 e^{i\delta} & c_1^0 c_2^0 c_3^0 - s_2^0 s_3^0 e^{i\delta} \end{pmatrix} + \epsilon_1 \epsilon_3 \begin{pmatrix} 0 & c_1^0 s_3^0 & -c_1^0 c_3^0 \\ 0 & c_2^0 s_1^0 s_3^0 & -c_2^0 c_3^0 s_1^0 \\ 0 & s_1^0 s_2^0 s_3^0 & -c_3^0 s_1^0 s_2^0 \end{pmatrix} \\
&+ \mathcal{O}(\epsilon_i^3),
\end{aligned}
\tag{12}
$$

where $s_i^0 = \sin\theta_i^0$ and $c_i^0 = \cos\theta_i^0$.

The rephasing invariant quantity J [21] given by

$$J = \text{Im}(V_{11}V_{22}V_{12}^*V_{21}^*) = s_1^2 s_2 s_3 c_1 c_2 c_3 \sin\delta \tag{13}$$

is independent of phase convention, making it important when discussing CP violation. Expanding J with ϵ_i to the second order gives

$$
\begin{aligned}
J =&\ J_0\Big(1 + \epsilon_1(3\cot 2\theta_1^0 + \csc 2\theta_1^0) + 2\epsilon_2\cot 2\theta_2^0 + 2\epsilon_3\cot 2\theta_3^0 + \frac{1}{4}\epsilon_1^2(9\cos 2\theta_1^0 - 5)\csc^2\theta_1^0 \\
&- 2\epsilon_2^2 - 2\epsilon_3^2 + 2\epsilon_1\epsilon_2(3\cos 2\theta_1^0 + 1)\cot 2\theta_2^0\csc 2\theta_1^0 + 4\epsilon_2\epsilon_3\cot 2\theta_2^0\cot 2\theta_3^0 \\
&+ 2\epsilon_1\epsilon_3(3\cos 2\theta_1^0 + 1)\cot 2\theta_3^0\csc 2\theta_1^0\Big) + \mathcal{O}(\epsilon_i^3) ,
\end{aligned}
\tag{14}
$$

in which $J_0 = (s_1^0)^2 s_2^0 s_3^0 c_1^0 c_2^0 c_3^0 \sin\delta$.

The general form Eqs. (12) and (14) look quite complicated since they are simple expansions in mathematics. We can simplify the general expansion by taking $\theta_3^0 = 0$ because θ_3^0 is small in both quark and lepton sectors. In this case, the result is

$$
\begin{aligned}
V =&\ \begin{pmatrix} c_1^0 & -s_1^0 & 0 \\ s_1^0 c_2^0 & c_1^0 c_2^0 & s_2^0 e^{i\delta} \\ s_1^0 s_2^0 & c_1^0 s_2^0 & -c_2^0 e^{i\delta} \end{pmatrix} + \epsilon_1 \begin{pmatrix} -s_1^0 & -c_1^0 & 0 \\ c_1^0 c_2^0 & -c_2^0 s_1^0 & 0 \\ c_1^0 s_2^0 & s_1^0 s_2^0 & 0 \end{pmatrix} + \epsilon_2 \begin{pmatrix} 0 & 0 & 0 \\ -s_1^0 s_2^0 & -c_1^0 s_2^0 & c_2^0 e^{i\delta} \\ c_2^0 s_1^0 & c_1^0 c_2^0 & s_2^0 e^{i\delta} \end{pmatrix} \\
&+ \epsilon_3 \begin{pmatrix} 0 & 0 & -s_1^0 \\ 0 & -s_2^0 e^{i\delta} & c_1^0 c_2^0 \\ 0 & c_2^0 e^{i\delta} & c_1^0 s_2^0 \end{pmatrix} + \frac{1}{2}\epsilon_1^2 \begin{pmatrix} -c_1^0 & s_1^0 & 0 \\ -c_2^0 s_1^0 & -c_1^0 c_2^0 & 0 \\ -s_1^0 s_2^0 & -c_1^0 s_2^0 & 0 \end{pmatrix} + \frac{1}{2}\epsilon_2^2 \begin{pmatrix} 0 & 0 & 0 \\ -c_2^0 s_1^0 & -c_1^0 c_2^0 & -s_2^0 e^{i\delta} \\ -s_1^0 s_2^0 & -c_1^0 s_2^0 & c_2^0 e^{i\delta} \end{pmatrix} \\
&+ \frac{1}{2}\epsilon_3^2 \begin{pmatrix} 0 & s_1^0 & 0 \\ 0 & -c_1^0 c_2^0 & -s_2^0 e^{i\delta} \\ 0 & -c_1^0 s_2^0 & c_2^0 e^{i\delta} \end{pmatrix} + \epsilon_1\epsilon_2 \begin{pmatrix} 0 & 0 & 0 \\ -c_1^0 s_2^0 & s_1^0 s_2^0 & 0 \\ c_1^0 c_2^0 & -c_2^0 s_1^0 & 0 \end{pmatrix} + \epsilon_2\epsilon_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -c_2^0 e^{i\delta} & -c_1^0 s_2^0 \\ 0 & -s_2^0 e^{i\delta} & c_1^0 c_2^0 \end{pmatrix} \\
&+ \epsilon_1\epsilon_3 \begin{pmatrix} 0 & 0 & -c_1^0 \\ 0 & 0 & -c_2^0 s_1^0 \\ 0 & 0 & -s_1^0 s_2^0 \end{pmatrix} + \mathcal{O}(\epsilon_i^3) ,
\end{aligned}
\tag{15}
$$

and the Jarlskog parameter reduces to

$$J = \sin\delta\Big(\epsilon_3 c_1^0 c_2^0 (s_1^0)^2 s_2^0 + \epsilon_2\epsilon_3 c_1^0 (s_1^0)^2 \cos 2\theta_2^0 - \frac{1}{8}\epsilon_1\epsilon_3 \sin 2\theta_2^0 (s_1^0 - 3\sin 3\theta_1^0)\Big) . \tag{16}$$

These expressions for mixing matrix and Jarlskog parameter are still complicated, making it difficult to capture physical meanings from them. A good choice of the zeroth order matrix V_0 will simplify the parametrization greatly, lead to fast convergency of the expansion, reflect the physical insight of a parametrization, and provide hints for underlying theory producing the mixing. Therefore, in the following two sections, different basis matrices will be applied to make the expansions simpler and useful for both theoretical and experimental analysis.

III. PARAMETRIZATION OF KM MATRIX WITH UNIT AND BIMAXIMAL BASIS MATRICES

A. The triminimal expansion

Since V_{CKM} is close to the unit matrix shown by Eq. (3), we can naturally set

$$\theta_1^{0Q} = \theta_2^{0Q} = \theta_3^{0Q} = 0 , \tag{17}$$

and consequently have

$$\epsilon_1^Q = \theta_1^Q = 0.2273^{+0.0011}_{-0.0003}, \quad \epsilon_2^Q = \theta_2^Q = 0.0383^{+0.0011}_{-0.0010}, \quad \epsilon_3^Q = \theta_3^Q = 0.0154^{+0.0008}_{-0.0006} , \tag{18}$$

which show that $(\epsilon_1^Q)^2 \sim \epsilon_2^Q \sim \epsilon_3^Q$. In most cases the approximation to the second order of ϵ_i is enough. However, in order to keep the consistency of magnitudes in the expansion, we display all terms of $\mathcal{O}((\epsilon_1^Q)^3)$ in our parametrization,

which is given by

$$V_{\mathrm{CKM}} = \begin{pmatrix} 1 - \frac{(\epsilon_1^Q)^2}{2} & \epsilon_1^Q - \frac{(\epsilon_1^Q)^3}{6} & e^{-i\delta}\epsilon_1^Q\epsilon_3^Q \\ \frac{(\epsilon_1^Q)^3}{6} - \epsilon_1^Q & 1 - \frac{(\epsilon_1^Q)^2}{2} & \epsilon_2^Q + e^{-i\delta}\epsilon_3^Q \\ \epsilon_1^Q\epsilon_2^Q & -\epsilon_2^Q - e^{i\delta}\epsilon_3^Q & 1 \end{pmatrix} + \mathcal{O}((\epsilon_1^Q)^4) \,, \tag{19}$$

where the rephasing of quark fields

$$c \to ce^{i\pi}, \quad s \to se^{i\pi}, \quad b \to be^{i(\pi+\delta)} \tag{20}$$

is implied to make the lowest order be unit matrix.

Before moving on to lepton sector, we need to talk about the QLC relations in Eq. (10) here. In terms of triminimal parameters, it is natural to rewrite Eq. (10) as

$$\theta_1^{0Q} + \theta_1^{0L} = \frac{\pi}{4}, \quad \theta_2^{0Q} + \theta_2^{0L} = \frac{\pi}{4}, \quad \theta_3^{0Q} = \theta_3^{0L} = 0, \tag{21}$$

and recognize all ϵ_i^Q and ϵ_i^L as small deviations. Therefore, corresponding to the choice in quark sector Eq. (17), we have

$$\theta_1^{0L} = \theta_2^{0L} = \frac{\pi}{4}, \quad \theta_3^{0L} = 0 \,, \tag{22}$$

in lepton sector. Similar redefinition of lepton fields

$$\mu \to \mu e^{i\pi}, \quad \nu^\mu \to \nu^\mu e^{i\pi}, \quad \nu^\tau \to \nu^\tau e^{i(\pi+\delta)} \tag{23}$$

is also implied to adjust the phases. To the second order of ϵ_i^L, lepton mixing matrix is expanded as

$$
\begin{aligned}
U_{\mathrm{PMNS}} =\ & \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} + \epsilon_1^L \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} + \epsilon_2^L \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \end{pmatrix} + \epsilon_3^L \begin{pmatrix} 0 & 0 & \frac{e^{-i\delta}}{\sqrt{2}} \\ 0 & -\frac{e^{i\delta}}{\sqrt{2}} & \frac{e^{-i\delta}}{2} \\ 0 & -\frac{e^{i\delta}}{\sqrt{2}} & -\frac{e^{-i\delta}}{2} \end{pmatrix} \\[2mm]
+\ & \frac{(\epsilon_1^L)^2}{2} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} + \frac{(\epsilon_2^L)^2}{2} \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \end{pmatrix} + \frac{(\epsilon_3^L)^2}{2} \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{2} & -\frac{1}{\sqrt{2}} \end{pmatrix} \\[2mm]
+\ & \frac{\epsilon_1^L\epsilon_2^L}{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} + \epsilon_2^L\epsilon_3^L \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{e^{i\delta}}{\sqrt{2}} & -\frac{e^{-i\delta}}{2} \\ 0 & \frac{e^{i\delta}}{\sqrt{2}} & -\frac{e^{-i\delta}}{2} \end{pmatrix} + \epsilon_1^L\epsilon_3^L \begin{pmatrix} 0 & 0 & \frac{e^{-i\delta}}{\sqrt{2}} \\ 0 & 0 & -\frac{e^{-i\delta}}{2} \\ 0 & 0 & \frac{e^{-i\delta}}{2} \end{pmatrix} + \mathcal{O}((\epsilon_i^L)^3) \,.
\end{aligned} \tag{24}
$$

As we expect, the zeroth order is the bimaximal matrix U_{bi}.

B. Wolfenstein-like parametrization

We now compare the triminimal parametrization with the Wolfenstein parametrization in quark sector. Since the original Wolfenstein parametrization takes the same phase convention as the standard CK form [20], which implies different choice of parameters, especially the CP-violating phase δ, it is complicated to arrive at the Wolfenstein parametrization from the triminimal expansion of KM matrix by adjusting the phases of the fields. Therefore, we only keep the original Wolfenstein expanding parameter λ, which satisfies $\lambda = \sin\theta_1^Q \approx \epsilon_1^Q - \frac{(\epsilon_1^Q)^3}{6}$, and introduce two new parameters with

$$f\lambda^2 = \sin\theta_2^Q \approx \epsilon_2^Q, \quad h\lambda^2 = \sin\theta_3^Q \approx \epsilon_3^Q \,. \tag{25}$$

The CP-violating phase δ has clear physical meaning so we naturally keep it. Finally, by substituting them into Eq. (19), we obtain a new Wolfenstein-like parametrization, given by

$$V_{\mathrm{CKM}} = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & e^{-i\delta}h\lambda^3 \\ -\lambda & 1 - \frac{\lambda^2}{2} & (f + e^{-i\delta}h)\lambda^2 \\ f\lambda^3 & -(f + e^{i\delta}h)\lambda^2 & 1 \end{pmatrix} \,, \tag{26}$$

which is a new simple form of quark mixing matrix given in our recent work [22]. Direct calculation of $f\lambda^3 = |V_{td}|$, $h\lambda^3 = |V_{ub}|$ and $|f + e^{-i\delta^Q}h|\lambda^2 = |V_{cb}|$ with the latest data (3) and (7) gives

$$\lambda = 0.2253 \pm 0.0007, \quad h = 0.303^{+0.014}_{-0.010}, \quad f = 0.754^{+0.022}_{-0.018}, \quad \delta^Q = 90.97^{\circ+2.77^\circ}_{-4.44^\circ}, \tag{27}$$

which are slightly different from the results in Ref. [22] where previous data are used. This new form of quark mixing matrix preserves the hierarchical structure of the mixing. More importantly, it is convenient for numerical analysis, especially for constraint of the CP-violating phase. Along with unitarity boomerangs it may be useful to study the presence of new physics.

To get unified Wolfenstein-like parametrizations for quark and lepton mixing, we need to use the QLC relations Eq. (10), of which the first two equations lead to the choice of the parameters in the lepton sector as

$$\theta_1^L = \pi/4 - \arcsin\lambda, \quad \theta_2^L = \pi/4 - \arcsin f\lambda^2, \tag{28}$$

i.e., parameters λ and f, which we introduce in quark sector are also employed in lepton sector. The other two parameters are one angle parameter related to θ_3^L and a CP-violating phase δ^L. Since the experimental data for the small angle θ_3^L is not accurate enough, we can either set $\theta_3^L = \eta\lambda$ or $\theta_3^L = \eta'\lambda^2$ depending on the value of $|U_{e3}|$. We now discuss these two cases separately.

Case 1: $\theta_3^L = \eta\lambda$

The lepton mixing matrix can be expanded in order of λ as

$$
U_{\text{PMNS}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{e^{-i\delta}\eta}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} - \frac{e^{i\delta}\eta}{\sqrt{2}} & \frac{1}{2}e^{-i\delta}\eta \\ -\frac{1}{2} & -\frac{e^{i\delta}\eta}{\sqrt{2}} - \frac{1}{2} & -\frac{1}{2}e^{-i\delta}\eta \end{pmatrix}
$$

$$
+ \lambda^2 \begin{pmatrix} -\frac{1}{2\sqrt{2}} & -\frac{\eta^2}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} & -\frac{e^{-i\delta}\eta}{\sqrt{2}} \\ \frac{1}{4} - \frac{f}{2} & -\frac{\eta^2}{4} + \frac{f}{2} - \frac{1}{4} & -\frac{\eta^2}{2\sqrt{2}} + \frac{1}{2}e^{-i\delta}\eta - \frac{f}{\sqrt{2}} \\ -\frac{f}{2} - \frac{1}{4} & \frac{\eta^2}{4} + \frac{f}{2} + \frac{1}{4} & -\frac{(\eta^2}{2\sqrt{2}} - \frac{1}{2}e^{-i\delta}\eta + \frac{f}{\sqrt{2}} \end{pmatrix}
$$

$$
+ \lambda^3 \begin{pmatrix} 0 & \frac{\eta^2}{2\sqrt{2}} & -\frac{e^{-i\delta}\eta^3}{6\sqrt{2}} - \frac{e^{-i\delta}\eta'}{2\sqrt{2}} \\ \frac{f}{2} & \frac{e^{i\delta}\eta^3}{6\sqrt{2}} - \frac{\eta^2}{4} + \frac{e^{i\delta}f\eta}{\sqrt{2}} + \frac{f}{2} & -\frac{1}{12}e^{-i\delta}\eta^3 - \frac{1}{4}e^{-i\delta}\eta + \frac{1}{2}e^{-i\delta}f\eta \\ \frac{f}{2} & \frac{e^{i\delta}\eta^3}{6\sqrt{2}} + \frac{\eta^2}{4} - \frac{e^{i\delta}f\eta}{\sqrt{2}} + \frac{f}{2} & \frac{1}{12}e^{-i\delta}\eta^3 + \frac{1}{4}e^{-i\delta}\eta + \frac{1}{2}e^{-i\delta}f\eta \end{pmatrix} + \mathcal{O}(\lambda^4). \tag{29}
$$

With the modulus of the element U_{e3} in Eq. (5) and the results for λ and f in Eq. (27), the new parameter is constrained by $0 < \eta < 1.923$.

Case 2: $\theta_3^L = \eta'\lambda^2$

In this case the expansion looks simpler since η' only starts to appear in $\mathcal{O}(\lambda^2)$ terms:

$$
U_{\text{PMNS}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{e^{-i\delta}\eta'}{\sqrt{2}} \\ \frac{1}{4} - \frac{f}{2} & \frac{f}{2} - \frac{e^{i\delta}\eta'}{\sqrt{2}} - \frac{1}{4} & \frac{1}{2}e^{-i\delta}\eta' - \frac{f}{\sqrt{2}} \\ -\frac{f}{2} - \frac{1}{4} & \frac{f}{2} - \frac{e^{i\delta}\eta'}{\sqrt{2}} + \frac{1}{4} & \frac{f}{\sqrt{2}} - \frac{1}{2}e^{-i\delta}\eta' \end{pmatrix}
$$

$$
+ \lambda^3 \begin{pmatrix} 0 & 0 & -\frac{e^{-i\delta}\eta'}{\sqrt{2}} \\ \frac{f}{2} & \frac{f}{2} & \frac{1}{2}e^{-i\delta}\eta' \\ \frac{f}{2} & \frac{f}{2} & -\frac{1}{2}e^{-i\delta}\eta' \end{pmatrix} + \mathcal{O}(\lambda^4). \tag{30}
$$

However, we have a larger range in this case with $0 < \eta' < 7.912$.

To the lowest order the Jarlskog parameter in both quark and lepton sectors is given by

$$J^Q = \text{Im}(V_{us}V_{cb}V_{ub}^*V_{cs}^*) = fh\lambda^6 \sin\delta^Q,$$

$$J^L = \text{Im}(U_{e2}U_{\mu3}U_{e3}^*U_{\mu2}^*) = \frac{1}{4\sqrt{2}}\eta\lambda\sin\delta^L,$$

$$J^L = \text{Im}(U_{e2}U_{\mu3}U_{e3}^*U_{\mu2}^*) = \frac{1}{4\sqrt{2}}\eta'\lambda^2\sin\delta^L. \tag{31}$$

The last two equations correspond to the two cases respectively.

IV.   PARAMETRIZATION OF THE KM MATRIX IN TRI-BIMAXIMAL PATTERN

A.   The triminimal expansion

Present data of the PMNS matrix indicate that the mixing in the lepton sector is closer to the tri-bimaximal mixing. One would have a much faster convergent expansion if it starts with a tri-bimaximal mixing form as the basis, which implies again the rephasing of lepton fields in Eq. (23) and the choice of θ_i^{0L} as

$$\theta_1^{0L} = \arcsin\frac{1}{\sqrt{3}}, \quad \theta_2^{0L} = \frac{\pi}{4}, \quad \theta_3^{0L} = 0 \ . \tag{32}$$

Under such conditions, the lepton mixing matrix expanded to the second order of ϵ_i^L is given by

$$
\begin{aligned}
U_{\text{PMNS}} =& \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} + \epsilon_1^L \begin{pmatrix} -\frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 0 \end{pmatrix} + \epsilon_2^L \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \\
&+ \epsilon_3^L \begin{pmatrix} 0 & 0 & \frac{e^{-i\delta}}{\sqrt{3}} \\ 0 & -\frac{e^{i\delta}}{\sqrt{2}} & \frac{e^{-i\delta}}{\sqrt{3}} \\ 0 & -\frac{e^{i\delta}}{\sqrt{2}} & -\frac{e^{-i\delta}}{\sqrt{3}} \end{pmatrix} + \frac{(\epsilon_1^L)^2}{2} \begin{pmatrix} -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix} + \frac{(\epsilon_2^L)^2}{2} \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \\
&+ \frac{(\epsilon_3^L)^2}{2} \begin{pmatrix} 0 & -\frac{1}{\sqrt{3}} & 0 \\ 0 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} + \epsilon_1^L \epsilon_2^L \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 0 \end{pmatrix} + \epsilon_2^L \epsilon_3^L \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{e^{i\delta}}{\sqrt{2}} & -\frac{e^{-i\delta}}{\sqrt{3}} \\ 0 & \frac{e^{i\delta}}{\sqrt{2}} & -\frac{e^{-i\delta}}{\sqrt{3}} \end{pmatrix} \\
&+ \epsilon_1^L \epsilon_3^L \begin{pmatrix} 0 & 0 & \sqrt{\frac{2}{3}}e^{-i\delta} \\ 0 & 0 & -\frac{e^{-i\delta}}{\sqrt{6}} \\ 0 & 0 & \frac{e^{-i\delta}}{\sqrt{6}} \end{pmatrix} + \mathcal{O}((\epsilon_i^L)^3) \ .
\end{aligned}
\tag{33}
$$

With Eq. (21) as our guide, we get the zeroth order of mixing angles in the quark sector as

$$\theta_1^{0Q} = \arcsin\frac{\sqrt{2}-1}{\sqrt{6}}, \ \theta_2^{0Q} = 0, \ \theta_3^{0Q} = 0 \ . \tag{34}$$

In this case the deviations are

$$\epsilon_1^Q = 0.0574^{+0.0010}_{-0.0003}, \quad \epsilon_2^Q = 0.0383^{+0.0011}_{-0.0010}, \quad \epsilon_3^Q = 0.0154^{+0.0008}_{-0.0006} \ , \tag{35}$$

according to Eq. (4). Thus $\epsilon_i^Q \sim \mathcal{O}(10^{-2})$, and we have a faster convergent expansion for quark mixing as

$$
\begin{aligned}
V_{\text{CKM}} =& \begin{pmatrix} \frac{\sqrt{2}+1}{\sqrt{6}} & \frac{\sqrt{2}-1}{\sqrt{6}} & 0 \\ -\frac{\sqrt{2}-1}{\sqrt{6}} & \frac{\sqrt{2}+1}{\sqrt{6}} & 0 \\ 0 & 0 & 1 \end{pmatrix} + \epsilon_1^Q \begin{pmatrix} -\frac{\sqrt{2}-1}{\sqrt{6}} & \frac{\sqrt{2}+1}{\sqrt{6}} & 0 \\ -\frac{\sqrt{2}+1}{\sqrt{6}} & -\frac{\sqrt{2}-1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \epsilon_2^Q \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \frac{\sqrt{2}-1}{\sqrt{6}} & -\frac{\sqrt{2}+1}{\sqrt{6}} & 0 \end{pmatrix} \\
&+ \epsilon_3^Q \begin{pmatrix} 0 & 0 & \frac{\sqrt{2}-1}{\sqrt{6}}e^{-i\delta} \\ 0 & 0 & \frac{\sqrt{2}+1}{\sqrt{6}}e^{-i\delta} \\ 0 & -e^{i\delta} & 0 \end{pmatrix} + \frac{(\epsilon_1^Q)^2}{2} \begin{pmatrix} -\frac{\sqrt{2}+1}{\sqrt{6}} & -\frac{\sqrt{2}-1}{\sqrt{6}} & 0 \\ \frac{\sqrt{2}-1}{\sqrt{6}} & -\frac{\sqrt{2}+1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{(\epsilon_2^Q)^2}{2} \begin{pmatrix} 0 & 0 & 0 \\ \frac{\sqrt{2}-1}{\sqrt{6}} & -\frac{\sqrt{2}+1}{\sqrt{6}} & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
&+ \frac{(\epsilon_3^Q)^2}{2} \begin{pmatrix} 0 & -\frac{\sqrt{2}-1}{\sqrt{6}} & 0 \\ 0 & -\frac{\sqrt{2}+1}{\sqrt{6}} & 0 \\ 0 & 0 & -1 \end{pmatrix} + \epsilon_1^Q \epsilon_2^Q \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\sqrt{2}+1}{\sqrt{6}} & \frac{\sqrt{2}-1}{\sqrt{6}} & 0 \end{pmatrix} + \epsilon_2^Q \epsilon_3^Q \begin{pmatrix} 0 & 0 & 0 \\ 0 & -e^{i\delta} & 0 \\ 0 & 0 & -\frac{\sqrt{2}+1}{\sqrt{6}}e^{-i\delta} \end{pmatrix} \\
&+ \epsilon_1^Q \epsilon_3^Q \begin{pmatrix} 0 & 0 & \frac{\sqrt{2}+1}{\sqrt{6}}e^{-i\delta} \\ 0 & 0 & -\frac{\sqrt{2}-1}{\sqrt{6}}e^{-i\delta} \\ 0 & 0 & 0 \end{pmatrix} + \mathcal{O}((\epsilon_i^Q)^3) \ .
\end{aligned}
\tag{36}
$$

Consistent with the results in Ref. [14], the QLC relations relate the tri-bimaximal matrix in the lepton sector with V_0' in the quark sector, noting that they are both more close to experimental data compared with the bimaximal matrix and unit matrix.

B. Wolfenstein-like parametrization

Similarly with Sec. III, we now discuss the corresponding Wolfenstein-like parametrizations in both quark sector and lepton sector in a unified way with the help of QLC relations. For quark mixing, we can employ $\rho = \epsilon_1^Q$ as expanding parameter and introduce two coefficients with $s\rho = \epsilon_2^Q$, $t\rho = \epsilon_3^Q$, i.e.,

$$\theta_1^Q = \arcsin\frac{\sqrt{2}-1}{\sqrt{6}} + \rho, \quad \theta_2^Q = s\rho, \quad \theta_3^Q = t\rho \,. \tag{37}$$

Since the expansion around basis V_0' converges faster than the case in Sec. III, it is accurate enough to calculate to $\mathcal{O}(\rho^2)$, given by

$$
\begin{aligned}
V_{\mathrm{CKM}} =& \begin{pmatrix} \frac{\sqrt{2}+1}{\sqrt{6}} & \frac{\sqrt{2}-1}{\sqrt{6}} & 0 \\ -\frac{\sqrt{2}-1}{\sqrt{6}} & \frac{\sqrt{2}+1}{\sqrt{6}} & 0 \\ 0 & 0 & 1 \end{pmatrix} + \rho \begin{pmatrix} -\frac{\sqrt{2}-1}{\sqrt{6}} & \frac{\sqrt{2}+1}{\sqrt{6}} & \frac{(\sqrt{2}-1)e^{-i\delta}t}{\sqrt{6}} \\ -\frac{\sqrt{2}+1}{\sqrt{6}} & -\frac{\sqrt{2}-1}{\sqrt{6}} & s + \frac{(\sqrt{2}+1)e^{-i\delta}t}{\sqrt{6}} \\ \frac{(\sqrt{2}-1)s}{\sqrt{6}} & -\frac{(\sqrt{2}+1)s}{\sqrt{6}} - e^{i\delta}t & 0 \end{pmatrix} \\
&+ \rho^2 \begin{pmatrix} -\frac{\sqrt{2}+1}{2\sqrt{6}} & -\frac{(\sqrt{2}-1)(t^2+1)}{2\sqrt{6}} & \frac{(\sqrt{2}+1)e^{-i\delta}t}{\sqrt{6}} \\ \frac{(\sqrt{2}-1)(s^2+1)}{2\sqrt{6}} & -\frac{(\sqrt{2}+1)(s^2+t^2+1)}{2\sqrt{6}} - e^{i\delta}ts & -\frac{(\sqrt{2}-1)e^{-i\delta}t}{\sqrt{6}} \\ \frac{(\sqrt{2}+1)s}{\sqrt{6}} & \frac{(\sqrt{2}-1)s}{\sqrt{6}} & -\frac{(\sqrt{2}+1)e^{-i\delta}ts}{\sqrt{6}} - \frac{s^2}{2} - \frac{t^2}{2} \end{pmatrix} \\
&+ \mathcal{O}(\rho^3) \,.
\end{aligned}
\tag{38}
$$

By using the data for elements V_{ud}, V_{ub}, V_{td} and V_{cb} in Eq. (3), we get the ranges for the parameters with

$$\rho = 0.0574^{+0.0006}_{-0.0007}, \quad s = 0.667^{+0.020}_{-0.016}, \quad t = 0.268^{+0.013}_{-0.002}, \quad \delta^Q = 90.30^{\circ\,+2.85^\circ}_{\quad-4.57^\circ} \,. \tag{39}$$

As we can see here, the expanding parameter is $\rho \sim \mathcal{O}(10^{-2})$, making the expansion converge fast as we mentioned before, and indicating that V_0' is indeed a good choice of basis matrix.

For the corresponding Wolfenstein-like matrix in lepton sector, by using the QLC relations in Eq. (10), we have

$$\theta_1^L = \frac{\pi}{4} - \left(\arcsin\frac{\sqrt{2}-1}{\sqrt{6}} + \rho\right), \quad \theta_2^L = \frac{\pi}{4} - s\rho \,. \tag{40}$$

We still need to retain the CP-violating phase δ^L and to introduce a new parameter τ (or τ'). Here similar discussions as in Sec. III are needed depending on the value of $|U_{e3}|$.

Case 1: $\theta_3^L = \tau\rho$

To good accuracy, we expand the PMNS matrix to $\mathcal{O}(\rho^2)$ and obtain

$$
\begin{aligned}
U_{\mathrm{PMNS}} =& \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} + \rho \begin{pmatrix} \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} & \frac{e^{-i\delta}\tau}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} - \frac{s}{\sqrt{6}} & \frac{s}{\sqrt{3}} - \frac{e^{i\delta}\tau}{\sqrt{2}} + \frac{1}{\sqrt{6}} & \frac{e^{-i\delta}\tau}{\sqrt{3}} - \frac{s}{\sqrt{2}} \\ -\frac{s}{\sqrt{6}} - \frac{1}{\sqrt{3}} & \frac{s}{\sqrt{3}} - \frac{e^{i\delta}\tau}{\sqrt{2}} - \frac{1}{\sqrt{6}} & \frac{s}{\sqrt{2}} - \frac{e^{-i\delta}\tau}{\sqrt{3}} \end{pmatrix} \\
&+ \rho^2 \begin{pmatrix} -\frac{1}{\sqrt{6}} & -\frac{\tau^2}{2\sqrt{3}} - \frac{1}{2\sqrt{3}} & -\sqrt{\frac{2}{3}}e^{-i\delta}\tau \\ \frac{s^2}{2\sqrt{6}} + \frac{s}{\sqrt{3}} + \frac{1}{2\sqrt{6}} & -\frac{s^2}{2\sqrt{3}} + \frac{e^{i\delta}\tau s}{\sqrt{2}} + \frac{s}{\sqrt{6}} - \frac{\tau^2}{2\sqrt{3}} - \frac{1}{2\sqrt{3}} & -\frac{s^2}{2\sqrt{2}} + \frac{e^{-i\delta}\tau s}{\sqrt{3}} - \frac{\tau^2}{2\sqrt{2}} + \frac{e^{-i\delta}\tau}{\sqrt{6}} \\ -\frac{s^2}{2\sqrt{6}} + \frac{s}{\sqrt{3}} - \frac{1}{2\sqrt{6}} & \frac{s^2}{2\sqrt{3}} - \frac{e^{i\delta}\tau s}{\sqrt{2}} + \frac{s}{\sqrt{6}} + \frac{\tau^2}{2\sqrt{3}} + \frac{1}{2\sqrt{3}} & -\frac{s^2}{2\sqrt{2}} + \frac{e^{-i\delta}\tau s}{\sqrt{3}} - \frac{\tau^2}{2\sqrt{2}} - \frac{e^{-i\delta}\tau}{\sqrt{6}} \end{pmatrix} \\
&+ \mathcal{O}(\rho^3) \,.
\end{aligned}
\tag{41}
$$

The upper bound of $|U_{e3}|$ gives $0 < \tau < 7.493$.

Case 2: $\theta_3^L = \tau'\rho^2$

The expansion to $\mathcal{O}(\rho^2)$ is given by

$$
U_{\mathrm{PMNS}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} + \rho \begin{pmatrix} \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} & 0 \\ \frac{1}{\sqrt{3}} - \frac{s}{\sqrt{6}} & \frac{s}{\sqrt{3}} + \frac{1}{\sqrt{6}} & -\frac{s}{\sqrt{2}} \\ -\frac{s}{\sqrt{6}} - \frac{1}{\sqrt{3}} & \frac{s}{\sqrt{3}} - \frac{1}{\sqrt{6}} & \frac{s}{\sqrt{2}} \end{pmatrix}
$$

$$
+ \rho^2 \begin{pmatrix} -\frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} & \frac{e^{-i\delta}\tau'}{\sqrt{3}} \\ \frac{s^2}{2\sqrt{6}} + \frac{s}{\sqrt{3}} + \frac{1}{2\sqrt{6}} & -\frac{s^2}{2\sqrt{3}} + \frac{s}{\sqrt{6}} - \frac{e^{i\delta}\tau'}{\sqrt{2}} - \frac{1}{2\sqrt{3}} & \frac{e^{-i\delta}\tau'}{\sqrt{3}} - \frac{s^2}{2\sqrt{2}} \\ -\frac{s^2}{2\sqrt{6}} + \frac{s}{\sqrt{3}} - \frac{1}{2\sqrt{6}} & \frac{s^2}{2\sqrt{3}} + \frac{s}{\sqrt{6}} - \frac{e^{i\delta}\tau'}{\sqrt{2}} + \frac{1}{2\sqrt{3}} & -\frac{s^2}{2\sqrt{2}} - \frac{e^{-i\delta}\tau'}{\sqrt{3}} \end{pmatrix}
$$

$$
+ \mathcal{O}(\rho^3) \, . \tag{42}
$$

A larger range for the parameter is obtained as $0 < \tau' < 125.987$. If we would like to control the parameters to be $\mathcal{O}(10^{-1})$, the former case is a better choice. However, further neutrino oscillation experiments are needed to determine $|U_{e3}|$ and the parameters we adopt here.

The Jarlskog parameters in tri-bimaximal pattern are given by

$$
J^Q = \mathrm{Im}(V_{us}V_{cb}V_{ub}^*V_{cs}^*) = \frac{1}{6}\sqrt{\frac{1}{6}(3 - 2\sqrt{2})} st\rho^2 \sin\delta^Q,
$$

$$
J^L = \mathrm{Im}(U_{e2}U_{\mu3}U_{e3}^*U_{\mu2}^*) = \frac{\tau\rho\sin\delta^L}{3\sqrt{6}},
$$

$$
J^L = \mathrm{Im}(U_{e2}U_{\mu3}U_{e3}^*U_{\mu2}^*) = \frac{\tau'\rho^2\sin\delta^L}{3\sqrt{6}} \, . \tag{43}
$$

V. DISCUSSIONS AND CONCLUSIONS

In this paper, by using triminimal and Wolfenstein-like expansion techniques, we study in more detail the Kobayashi-Maskawa matrix in both quark and lepton sectors. Our motivation is based on the consideration of the convenience the KM form exhibits when discussing some problems such as unitarity boomerang and maximal CP violation.

In the previous two sections, we choose the unit matrix and V_0' as basis matrices for quark mixing, while bimaximal and tri-bimaximal matrices for lepton mixing. Naturally, a question arises here: which pattern is better, bimaximal, or tri-bimaximal? On one hand, the bimaximal matrix is related with the unit matrix through QLC relations. The corresponding triminimal and Wolfenstein-like expansions are both comparatively simple and symmetric. On the other hand, compared with bimaximal matrix, the tri-bimaximal matrix is closer to the experimental data; thus, the expansion based on it converges much faster. This can be reflected from the smallness of the triminimal angles and the Wolfenstein-like expanding parameter. Therefore, if we are interested only in the leading order contribution, then the tri-bimaximal matrix should be chosen. Theoretically, there have been some attempts at understanding both bimaximal and tri-bimaximal matrices with the introduction of new symmetries to the standard model fermions. However, we still need a more fundamental theory to decide which pattern should be used, to reveal the origin of the mixing matrices.

The Wolfenstein-like parametrizations presented in this paper are simpler than the results in Ref. [20] and can be transformed to triminimal expansions with relations among parameters. Here by using two sets of the Wolfenstein-like parameters, i.e., λ, f, h, η(or η'), δ^Q, and δ^L in Sec. III and ρ, s, t, τ(or τ'), δ^Q, and δ^L in Sec. IV, we unify the parametrization of the KM matrix in quark and lepton sectors. The parameters in the quark sector can be well determined with current experimental data. However, the experimental results in lepton sector, especially for $|U_{e3}|$ and the CP-violating phase δ^L, are far from enough.

The unified description of fermion mixing we get here results from the QLC relations in the KM form, and these relations are only approximately valid; thus, one may doubt the validity of the results. Actually, we can discuss this in a reversed way. By regarding these parameters for quarks and leptons as independent from each other, we can determine the parameters with data from quark experiments and lepton oscillations separately, and check the QLC relations. Even if the QLC relations violate, these parametrizations still stand separately in the quark sector and lepton sector. Thus, our study is helpful in understanding the mixing phenomenologically and may provide useful tools in searching for a profound theory on the fermion masses and mixing.

Acknowledgments

[1] N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963).
[2] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).
[3] B. Pontecorvo, Sov. Phys. JETP **6**, 429 (1957).
[4] Z. Maki, M. Nakagawa, and S. Sakata, Prog. Theor. Phys. **28**, 870 (1962).
[5] L.L. Chau and W.Y. Keung, Phys. Rev. Lett. **53**, 1802 (1984).
[6] P.H. Frampton and X.-G. He, Phys. Lett. **B688**, 67 (2010); P.H. Frampton and X.-G. He, Phys. Rev. **D82**, 017301 (2010).
[7] S.-W. Li and B.-Q. Ma, Phys. Lett. **B691**, 37 (2010).
[8] Y. Koide, Phys. Lett. **B607**, 123 (2005); Y. Koide and H. Nishiura, Phys. Rev. **D79**, 093005 (2009).
[9] D. Hochberg and R.G. Sachs, Phys. Rev. **D27**, 606 (1983); B. Stech, Phys. Lett. **B130**, 189 (1983); I. Dunietz, O.W. Greenberg and D.-d. Wu, Phys. Rev. Lett. **55**, 2935 (1985); Y. Koide, M. Nakahara and C. Hamzaoui, US-91-04, SU-LA-91-03.
[10] Particle Data Group, J. Phys. **G37**, 075021 (2010).
[11] M.C. Gonzalez-Garcia and M. Maltoni, Phys. Rept. **460**, 1 (2008).
[12] L. Wolfenstein, Phys. Rev. Lett. **51**, 1945 (1983).
[13] S. Pakvasa, W. Rodejohann and T.J. Weiler, Phy. Rev. Lett. **100**, 111801 (2008); X.-G. He, S.-W. Li and B.-Q. Ma, Phys. Rev. **D78**, 111301R (2008).
[14] S.-W. Li and B.-Q. Ma, Phys. Rev. **D77**, 093005 (2008).
[15] F. Vissani, hep-ph/9708483; V.D. Barger, S.Pakvasa, T.J. Weiler, and K. Whisnant, Phys. Lett. **B437**, 107 (1998); A.J. Baltz, A.S. Goldhaber, and M. Goldhaber, Phys. Rev. Lett. **81**, 5730 (1998); I. Stancu and D.V. Ahluwalia, Phys. Lett. **B460**, 431 (1999); H. Georgi and S.L. Glashow, Phy. Rev. **D61**, 097301 (2000); N. Li and B.-Q. Ma, Phys. Lett. B **600**, 248 (2004) [arXiv:hep-ph/0408235].
[16] P.F. Harrison, D.H. Perkins, and W.G. Scott, Phys. Lett. **B458**, 79 (1999); Phys. Lett. **B530**, 167 (2002); Z.Z. Xing, Phys. Lett. **B533**, 85 (2002); P.F. Harrison and W.G. Scott, Phys. Lett. **B535**, 163 (2002); Phys. Lett. **B557**, 76 (2003); X.-G. He and A. Zee, Phys. Lett. **B560**, 87 (2003); N. Li and B.-Q. Ma, Phys. Rev. D **71**, 017302 (2005) [arXiv:hep-ph/0412126]. See also L. Wolfenstein, Phys. Rev. **D18**, 958 (1978); Y. Yamanaka, H. Sugawara, and S. Pakvasa, Phys. Rev. **D25**, 1895(1982); **D29**, 2135(E)(1984).
[17] N. Li and B.-Q. Ma, Eur. Phys. J. **C42**, 17 (2005).
[18] A. Y. Smirnov, in *2nd International Workshop on Neutrino Oscillations in Venice (NOVE) December 3-5, 2003, Venice, Italy*, published in Venice 2003, Neutrino Oscillations, p. 1; H. Minakata and A.Y. Smirnov, Phys. Rev. **D70**, 073009 (2004); M. Raidal, Phys. Rev. Lett. **93**, 161801 (2004); P.H. Frampton and R.N. Mohapatra, JHEP **0501**, 025 (2005); N. Li and B.-Q. Ma, Phys. Rev. **D71**, 097301 (2005); J. Ferrandis and S. Pakvasa, Phys. Rev. **D71**, 033004 (2005); S.K. Kang, C.S. Kim and J. Lee, Phys. Lett. **B619**, 129 (2005); S. Antusch, S.F. King and R.N. Mohapatra, Phys. Lett. **B618**, 150 (2005); M.A. Schmidt and A.Y. Smirnov, Phys. Rev. **D74**, 113003 (2006); F. Plentinger, G. Seidl and W. Winter, Phys. Rev. **D76**, 113003 (2007).
[19] Y.-J. Zheng, Phys. Rev. **D81**, 073009 (2010).
[20] X.-G. He, S.-W. Li and B.-Q. Ma, Phys. Rev. **D79**, 073001 (2009).
[21] C. Jarlskog, Phys. Rev. Lett. **55**, 1039 (1985); D.-d. Wu, Phys. Rev. D **33**, 860 (1986); O.W. Greenberg, Phys. Rev. **D32**, 1841 (1985).
[22] N. Qin and B.-Q. Ma, Phys. lett **B695**, 194 (2011).